

Fredrickson-Andersen model on Bethe lattice with random pinning

HARUKUNI IKEDA¹ and KUNIMASA MIYAZAKI¹

¹ *Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

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Abstract – We study the effects of random pinning on the Fredrickson-Andersen model on the Bethe lattice. We find that the nonergodic transition temperature rises as the fraction of the pinned spins increases and the transition line terminates at a critical point. The freezing behavior of the spins is analogous to that of a randomly pinned p -spin mean-field spin glass model which has been recently reported. The diverging behavior of correlation lengths in the vicinity of the terminal critical point is found to be identical to the prediction of the inhomogeneous mode-coupling theory at the A_3 singularity point for the glass transition.

Introduction. – The nature of the glass transition is still elusive [1–3]. It is a major challenge to understand universal behavior of the transition, such as the non-Arrhenius growth of the viscosity, non-exponential relaxation of the correlation functions, and spatially heterogeneous dynamics of the fluctuations. The ultimate goal is to establish whether the transition is associated with any thermodynamic singularity, or it is caused by purely kinetic mechanism [1, 2]. Obvious reasons that hamper the progress of our understanding of the glass transition are the inhibitedly long time scales required to reach the transition point experimentally and the lack of an ideally simple finite-dimensional model which exhibits a true glass transition, if any.

Recently, a novel idea to bypass the difficulty to access the glass transition temperature by randomly freezing, or pinning, a fraction of degrees of freedom of the equilibrated system has been proposed [4–7]. Cammarota *et al.* have analyzed the effects of random pinning on the glass transition of the p -spin mean-field spin glass model [6, 7]. It was found that both the ideal glass transition temperature, T_K , and the dynamic transition temperature, T_d , rise as the fraction of the pinned spins, c , increases. Furthermore, the two transition lines, $T_K(c)$ and $T_d(c)$, are found to merge and terminate at a finite c . This end point is argued to be a critical point whose universality class is that of the mean-field random-field Ising model [8]. In the terminology of the mode-coupling theory (MCT), which is a mean-field dynamical theory of the glass transition, this

end point is characterized as the A_3 singularity, where the anomalous dynamical scalings, such as the logarithmic relaxation dynamics and distinct diverging length scales, are predicted [9]. Verification of the ideal glass transition of randomly pinned systems by experiments and simulations for realistic systems may be crucial to prove (or disprove) the very existence of the bona-fide glass transition point at finite dimensions [10,11]. At the same time, it is imminent to establish the relationship of thermodynamic scenarios such as the random first order transition theory (RFOT) with other kinetic scenarios which do not necessarily require thermodynamic singularities behind the glassy slow dynamics [2,12–14].

In this letter, we argue that the singular behavior of the glass transition by random pinning analogous to that of the p -spin mean-field spin glass model can be also observed for a purely kinetic model on the Bethe lattices (or random graphs). Kinetically constrained models (KCMs) are toy lattice or spin models in which Hamiltonian is free from the interaction but a non-trivial constraint is imposed on the dynamic rule [2, 14, 15]. Many KCMs are known to display slow dynamics in the collective movement of the particles or spins at low temperatures but they are of purely kinetic origin since their thermodynamics are trivially ideal-gas like. Their glassy behavior is genetically distinct from those of thermodynamic scenarios such as RFOT where thermodynamic singularities encoded in the free energy landscape play a pivotal role and escort the slow dynamics. Indeed, most KCMs show no glass transi-

tion at finite temperatures. Notable exceptions are KCMs on the Bethe lattices (or random graphs) [14, 16]. One of the KCMs called the Fredrickson-Andersen model (FAM) is known to undergo a nonergodic transition called the Bootstrap Percolation (BP) transition at a finite temperature T_d on the Bethe lattice [15–17]. It has been demonstrated that the time evolution of the order parameter of the FAM shows dynamical anomaly similar to those predicted by the MCT for structural glass formers, such as the two step relaxation and algebraic divergence of the relaxation time near T_d [16]. Here, we demonstrate that the randomly pinned FAM on the Bethe lattice also shows the analogous critical behavior as that of the randomly pinned p -spin mean-field spin glass model such as the A_3 higher order singularity [6, 7].

Model. – We consider a FAM on the Bethe lattice. In the absence of pinned spins, this model is defined as follows. We consider N Ising spins $\sigma_i \in \{-1, +1\}$, $i = 1, \dots, N$. The Hamiltonian of the system is free from the interaction and can be written as

$$H = -\frac{1}{2} \sum_{i=1}^N \sigma_i. \quad (1)$$

The equilibrium distribution of the up spins is given by

$$p = \frac{1}{1 + \exp(-1/T)}, \quad (2)$$

where p varies from $1/2$ to 1 , corresponding to $T = \infty$ and $T = 0$, respectively.

The spin variables represent coarse grained mobility fields. $\sigma_i = +1$ represents the immobile region and $\sigma_i = -1$ is the mobile region or defect [14]. The idea of the FAM is to introduce a kinetic constraint on the time evolution of the spins in such a way that flipping of a spin is more difficult when it is surrounded by many up spins (immobile regions). To be more specific, the i -th spin at each Monte-Carlo step can flip with a transition probability

$$w(\sigma_i \rightarrow -\sigma_i) = \min\{1, e^{-\sigma_i/T}\}, \quad (3)$$

only if the number of the nearest spins in the state -1 is larger than or equal to f [14, 15]. It is known that the FAM on a lattice in finite dimensions does not exhibit the transition at a finite temperature but it does so on the Bethe lattices and random graphs [14]. The FAM undergoes the nonergodic transition at a finite temperature if the connectivity k , or the number of the nearest neighbours $k+1$, satisfies $k > f > 1$ [16].

Now we consider to pin or freeze a fraction c of the spins randomly chosen from the N spins which are initially equilibrated at a temperature T before pinning. Note that pinning spins in the equilibrated configuration at T is essential in the following argument. The dynamic rule after pinning is basically the same as the bulk system; the i -th spin can flip with the probability of eq.(3), only if the

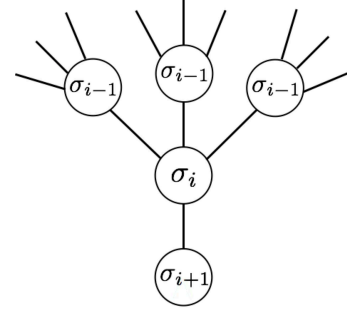


Fig. 1: The Bethe lattice with $k = 3$.

number of the nearest spins in the state -1 is larger than or equal to f and the i -th spin is not pinned. In the following, we only consider the case of $k = 3$ and $f = 2$ (see Fig. 1), but conclusions for different sets of (k, f) do not change qualitatively as long as $k > f > 1$.

Phase diagram. – Long-time behavior of the randomly pinned FAM on the Bethe lattice starting from the equilibrium distribution at the initial time can be evaluated analytically invoking the recursive relation on the lattice. We shall evaluate the persistent function ϕ , which is the total fraction of the dynamically frozen spins at the long-time limit, defined by

$$\phi = c + P_+ + P_-. \quad (4)$$

Here the first term c represents the fraction of the pinned spins and P_{\pm} represent the fraction, or the probability, of the spins ultimately arrested in the $+$ ($-$) state among the $(1 - c)N$ unpinned spins, respectively. P_+ is written as

$$P_+ = (1 - c)p \sum_{n=0}^{f-1} \binom{k+1}{n} B^n (1 - B)^{k+1-n}, \quad (5)$$

where p is given by eq.(2) and B is the conditional probability that a spin is in the state -1 or flipped down to the state -1 given that one of the nearest neighbours was arrested in the state $+1$. B obeys the following expression;

$$B = 1 - p + (1 - c)p \sum_{n=0}^{k-f} \binom{k}{n} B^{k-n} (1 - B)^n. \quad (6)$$

The first term, $1 - p$, on the right hand side of eq.(6) represents the equilibrium probability of -1 spins. The fact that it is independent of c is the reflection that the spins have been randomly pinned from the equilibrium distribution. The second term represents the probability of the spins originally at the state $+1$ but eventually flipped down to the state -1 . Eq. (6) is closely related to that of the multi-component extension of the FAM and associated models in Refs [18, 20]. Indeed, eq.(6) can be obtained by freezing one of the degree of freedom in the equation for the binary system considered in Ref [20] (see

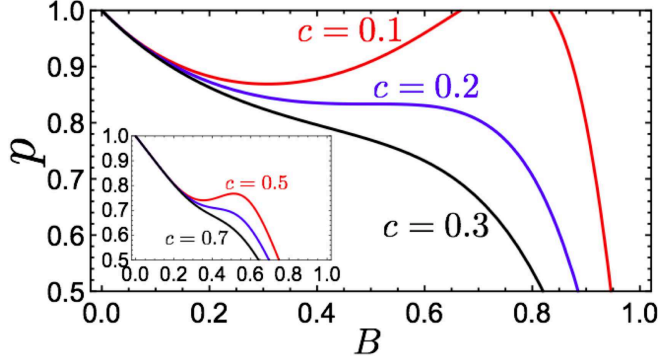


Fig. 2: p as a function of B for $k = 3$ and $f = 2$. Three lines correspond to $c = 0.1, 0.2$ and 0.3 from the top to the bottom. Inset: Same results for $k = 15$ and $f = 7$, where $c = 0.5, 0.6$ and 0.7 from the top to the bottom.

eq.(3) therein). Likewise, P_- can be written as

$$P_- = (1-c)(1-p) \sum_{n=0}^{f-1} \binom{k+1}{n} B'^n (1-B')^{k+1-n}, \quad (7)$$

where B' is the conditional probability that a spin is in the state -1 or flipped down to the state -1 given that one of the nearest neighbours was arrested in the state -1 , which is given by the solution of

$$B' = 1 - p + (1-c)p \sum_{n=0}^{k-f+1} \binom{k}{n} B^{k-n} (1-B)^n. \quad (8)$$

From eqs.(5) and (7), the total fraction of the arrested spins, eq.(4), is written as

$$\phi = c + (1-c) \left[p \Psi_{k+1}^f(B) + (1-p) \Psi_{k+1}^f(B') \right], \quad (9)$$

where we have defined an auxiliary function

$$\Psi_k^f(B) \equiv \sum_{n=0}^{f-1} \binom{k}{n} (1-B)^{k-n} B^n. \quad (10)$$

One observes that B' and ϕ can be computed from the self-consistent equation for B given by eq.(6). In the limit of $c = 0$, these expressions reduce to those studied in Ref. [16]. Eq. (9) obtained for the stationary state coincides with the long-time limit of the time-dependent persistent function $\phi(t)$ evaluated for the equilibrium initial configuration [14,16,18,19]. We checked this by the Monte-Carlo (MC) simulation for several state points (see the inset of Fig. 3). For $k = 3$ and $f = 2$, eq.(6) can be explicitly written as

$$B = 1 - p + (1-c)p \{ B^3 + 3B^2(1-B) \}. \quad (11)$$

If $c = 0$, there is always a trivial solution $B = 1$, *i.e.*, all spins can flip eventually and the system is ergodic. However, when the temperature is lowered from above, or

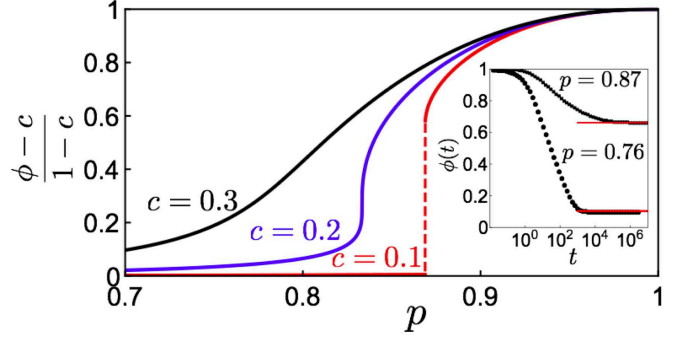


Fig. 3: p dependence of $(\phi(c,p) - c)/(1-c)$ for several c 's: The transition is discontinuous at $c < 1/5$ and continuous at $c = 1/5$. For $c > 1/5$, only the crossover is observed. Inset: The t -dependence of $\phi(t)$ for $c = 0.1$ and several p 's obtained by MC simulation (filled symbols). The solid lines are the solutions of the stationary equation, eq. (9).

p is raised from below from $p = 1/2$, B discontinuously jumps from 1 to $B_d = 1/4$ at a transition point, $p_d = 8/9$, and concomitantly ϕ jumps from 0 to a finite value of $\phi_d = 0.673$. This is the A_2 transition in the MCT terminology [3].

If $c \neq 0$, $B \neq 1$ always holds and eq.(11) can be solved immediately to obtain

$$p(B, c) = \frac{1-B}{1 - (1-c)[B^3 + 3B^2(1-B)]}. \quad (12)$$

In Fig. 2, we show $p(B, c)$ as a function of B for several c 's. If $c \neq 0$ but small, B -dependence of p shows a nonmonotonic behavior (see $c = 0.1$ of Fig. 2). When p is small (high temperature) where most of the spins are down (or mobile), there is only one solution for B close to 1, where a small fraction of the spins freezes in the vicinity of the pinned spins and most of the spins can still flip. As we increase p , the nontrivial branches at a small B (or a large ϕ) appear discontinuously at $p = p_d$, which is the signal of the A_2 singularity. The width of the gap of the discontinuous jump of B decreases as c is increased. As c increases further, the situation qualitatively changes. The minimum of $p(B, c)$ at B_d becomes unstable and eventually disappears. It is clear from eq. (12) that this happens at $c_s = 1/5$. At this point, the A_2 singularity disappears and the transition becomes continuous. This is the signal of the A_3 singularity. The MCT predicts that the critical behavior of the A_3 singularity is distinct from that of the A_2 singularity [3,6,7,9]. Qualitatively the same trend is observed for different sets of k and f (see the inset of Fig. 2).

$\phi(c, p)$ evaluated by numerically solving eqs. (8)–(11) is plotted in Fig. 3 for several c 's. The behavior of $\phi(c, p)$ is qualitatively the same as that of the nonergodic parameter evaluated from the MCT across the A_3 singularity point. Note that similar A_3 singularities have been analyzed for binary and ternary mixtures of the FAM on the Bethe lattices and random graphs [18–21].

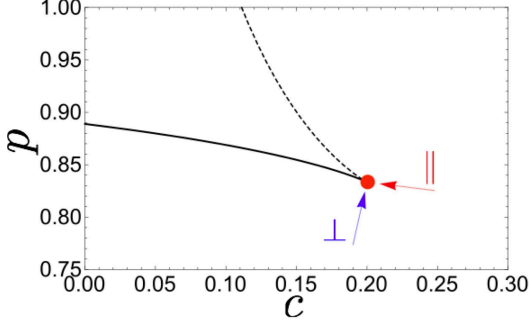


Fig. 4: The (dynamic) phase diagram of the randomly pinned FAM: The thick solid line is the nonergodic transition line $p_d(c)$. The dashed line is a “spinodal” line at which the larger branches of B disappear. The filled circle at the end of the thick solid line at $(c = 1/5, p = 5/6)$ is the terminal critical point of the higher order singularity. The arrows represent the directions of vectors \vec{e}_\perp and \vec{e}_\parallel (see the text).

The nonergodic transition line as a function of c , $p_d(c)$, is shown in Fig. 4 in the thick solid line. Recall that there exist three solutions of B of eq. (11) if p is slightly above p_d and the smallest B is the real solution. As it is observed in Fig. 2, if c is slightly smaller than c_s , there is always a finite p ($> p_d$) above which the larger solutions of B disappear. In Fig. 4, this “spinodal” line is shown in dashed line. This “phase-diagram” demonstrates that the transition point p_d decreases (or T_d increases) as c increases and eventually terminates at a critical point. The critical point can be evaluated analytically as $c_s = 1/5$ and $p_s = p_d(c_s) = 5/6$. At $c > c_s$, there is no distinction between the ergodic (fluid) and nonergodic (glassy) phases and ϕ continuously increases as p increases. The behavior of $p_d(c)$ in Fig. 4 is qualitatively the same as that evaluated for the randomly pinned p -spin mean-field spin glass model [6, 7]. We present the phase diagram for $k = 3$ and $f = 2$ but the results remain unchanged for other sets of k and f , as long as $k > f > 1$.

Higher Order Singularity. – If $c < c_s$, the transition across the thick solid line in Fig. 4 is discontinuous and the order parameters, ϕ (or B), behave as $\phi - \phi_d$ (or $B - B_d$) $\propto |p_d - p|^{1/2}$ [16]. At $c = c_s = 1/5$, however, the properties of the transition qualitatively change. In the following, we show that the critical behavior around this terminal critical point is indeed characterized as the A_3 singularity [3, 9, 22, 23].

We first introduce a function defined by

$$Q(c, p, B) \equiv 1 - p + (1 - c)p \{B^3 + 3B^2(1 - B)\} - B. \quad (13)$$

From eq.(11), $Q = 0$ always holds. We shall expand this

function around the transition point as

$$Q(c, p, B) = \frac{\partial Q}{\partial B} \delta B + \frac{1}{2} \frac{\partial^2 Q}{\partial B^2} \delta B^2 + \frac{1}{3!} \frac{\partial^3 Q}{\partial B^3} \delta B^3 + \frac{\partial Q}{\partial c} \delta c + \frac{\partial Q}{\partial p} \delta p + \left(\frac{\partial^2 Q}{\partial c \partial B} \delta c + \frac{\partial^2 Q}{\partial p \partial B} \delta p \right) \delta B + \dots, \quad (14)$$

where $\delta B = B - B_d$ and $\delta c = c - c_d$. The transition point is called the A_2 singularity point if $\partial Q / \partial B = 0$ and $\partial^2 Q / \partial B^2 \neq 0$, and the A_3 singularity point if $\partial Q / \partial B = \partial^2 Q / \partial B^2 = 0$ and $\partial^3 Q / \partial B^3 \neq 0$ [9, 22, 23]. It is obvious that on the transition line, $p_d(c)$, of Fig. 4 for small c , $\partial Q / \partial B = 0$ and $\partial^2 Q / \partial B^2 \neq 0$ are satisfied and that they are of the A_2 type. However, exactly on the terminal critical point, $(c_s, p_s, B_s) = (1/5, 5/6, 1/2)$, $\partial^2 Q / \partial B^2$ vanishes and this point is categorized as the A_3 singularity point. In the vicinity of the terminal critical point, eq.(13) can be written as

$$0 \approx \frac{1}{12} \varepsilon_c + \frac{1}{2} \varepsilon_p + \left(\frac{3}{16} \varepsilon_c + \frac{3}{4} \varepsilon_p \right) \delta B - \frac{4}{3} \delta B^3, \quad (15)$$

where we defined $\varepsilon_c = (c_s - c)/c_s$ and $\varepsilon_p = (p_s - p)/p_s$. Note that we have two control parameters, $\vec{\varepsilon} = (\varepsilon_c, \varepsilon_p)$. In order to investigate the critical behavior, it is convenient to transform the set of control parameters parallel and perpendicular to the transition line. Since the first two terms of eq. (15) are written in the form of the product of $\vec{\varepsilon}$ and a constant vector $(1/12, 1/2)$, it is natural to introduce the unit vectors directing to and perpendicular to this constant vector by $\vec{e}_\perp = \frac{1}{\sqrt{37}}(1, 6)$ and $\vec{e}_\parallel = \frac{1}{\sqrt{37}}(-6, 1)$ (see arrows in Fig. 4), so that $\vec{\varepsilon}$ can be written as $\vec{\varepsilon} = \varepsilon_\perp \vec{e}_\perp + \varepsilon_\parallel \vec{e}_\parallel$. With this expression, eq.(15) can be rewritten as

$$0 = -\frac{4\delta B^3}{3} + \frac{\sqrt{37}\varepsilon_\perp}{12} + \frac{3\delta B(25\varepsilon_\perp - 2\varepsilon_\parallel)}{16\sqrt{37}}, \quad (16)$$

and one immediately finds $\delta B \approx |\varepsilon_\perp|^{1/3}$ if $\varepsilon_\perp \neq 0$. On the other hand, if $\varepsilon_\perp = 0$, or if the critical point is approached from the direction parallel to \vec{e}_\parallel , one finds $\delta B \approx (-\varepsilon_\parallel)^{1/2}$.

Correlation lengths. – It is established, in the context of the mean-field scenario of the glass transition, that the cooperative dynamics are associated with diverging correlation lengths near the dynamic glass transition point. These lengths are predicted by the inhomogeneous version of the MCT (IMCT) both for the A_2 and A_3 singularity points [9, 24–26]. In this section, we discuss the correlation lengths of the randomly pinned FAM, using a point-to-set function, which is a quantity introduced to analyze the spatial correlations in glassy systems [27, 28].

The local structure of the Bethe lattice is Cayley-tree-like as shown in Fig. 1. We freeze the degrees of freedom of the spins at the outermost branch σ_0 and set $B_0 = 0$. We consider how much the effect of the frozen boundary penetrates down to the inner branches. The value of B_i down from the 0-th node is written by the recursive equation as

$$B_{i+1} = 1 - p + (1 - c)p \{B_i^3 + 3B_i^2(1 - B_i)\}. \quad (17)$$

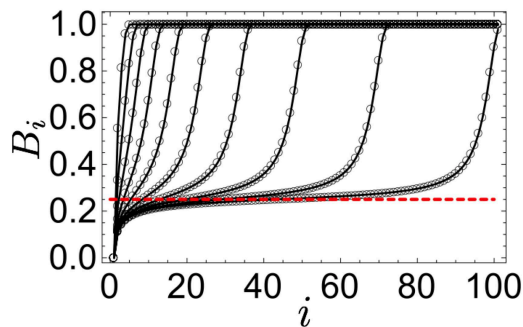


Fig. 5: Dependence of B_i on the depth of the inner branch for $c = 0$ and for various p 's. We define $(p_d - p)/p_d = 2^{-n}$ and the lines with circles correspond to $n = 1, 2, \dots, 10$ from left to right. The dashed horizontal line represents the threshold value $1/4$.

In Fig. 5, the numerical solution for B_i of eq.(17) for $c = 0$ is shown as a function of i for several values of p 's ($< p_d$) in ergodic states. B_i for $i > 0$ remains very small near the boundary but it relaxes to 1 far away from the boundary ($i \gg 1$). As p approaches to p_d from below, the distance over which the effect of the boundary penetrates grows. We shall define the correlation length ξ as the i -th point at which B_i exceeds the threshold whose value is set to B_d , the solution of eq.(11) at $p = p_d$ [29]. Numerical results for ξ for $c = 0, 0.1$, and 0.2 are shown as a function of the distance from the transition point $\varepsilon = (p_d - p)/p_d$ in Fig. 6. It is clearly seen that ξ 's behave as $|p_d - p|^{-1/2}$ for $c = 0$ and 0.1 [29]. But for $c = 1/5$, one observes $\xi \propto |p_d - p|^{-2/3}$. This asymptotic behavior of ξ can be analytically explained by the linear analysis of eq.(17) around p_d . Let us define the distance of B_i from the plateau value B_d by $\delta B_i \equiv B_i - B_d$. We assume that the variation of B_i is small and can be represented using a large number l as $\delta B_{i+1} = (1 - l^{-1})\delta B_i$. Since $\delta B_i \approx (1 - l^{-1})^i \approx e^{-i/l}$, l can be identified as ξ . Substituting this expression back to the recursive equation, eq. (17), we arrive at

$$\xi^{-1} \delta B_i = Q(c, p, B_i), \quad (18)$$

where $Q(c, p, B_i)$ is defined by eq.(13). Expanding $Q(c, p, B_i)$ in terms of δB_i around B_d like eq.(14), it is straightforward to show that ξ behaves as $\xi \propto \varepsilon^{-1/2}$ when $c < 1/5$. In the vicinity of the terminal critical point $(c_s, p_s) = (1/5, 5/6)$, the behavior of ξ qualitatively changes. If $\varepsilon_\perp \neq 0$, we have $\xi \propto \varepsilon_\perp^{-2/3}$, thus reproducing the results of Fig. 6. If $\varepsilon_\perp = 0$, one finds $\xi \propto \varepsilon_\parallel^{-1}$. The direction dependence of the growth of ξ is shown in Fig. 7 together with the direct numerical results. It is known that the exponents for the length scales of the model systems on the Bethe lattices are twice as large as the corresponding values of the mean-field theory [29–31]. Therefore, the exponent of the diverging length $\xi \sim \varepsilon^{-\nu}$ obtained above should correspond to the value of the mean-field theory of $\nu = 1/4$ instead of $1/2$ for $c < 1/5$. Likewise,

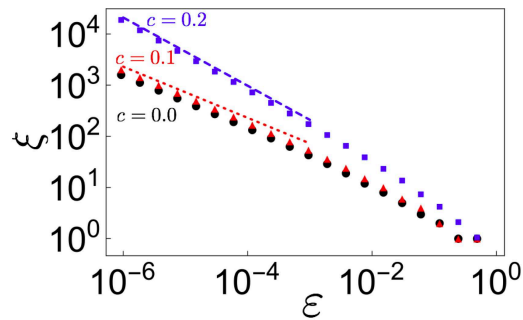


Fig. 6: ξ as a function of $\varepsilon = (p_d - p)/p_d$ for several c 's. Circles are for $c = 0$, triangles for $c = 0.1$, and squares are for $c = 0.2$. Thin dotted and dashed lines represent $|\varepsilon|^{-1/2}$ and $|\varepsilon|^{-2/3}$, respectively.

near the terminal critical point, the exponents should be $\nu_\perp = 1/3$ for $\varepsilon_\perp \neq 0$ and $\nu_\parallel = 1/2$ for $\varepsilon_\perp = 0$, respectively. These results are equivalent with those obtained from the IMCT [9, 26].

Conclusion. — In this letter, the stationary properties of the randomly pinned FAM on the Bethe lattice have been discussed. The order parameter, ϕ , or the probability that the spin flips, B , exhibits analogous behavior as the nonergodic parameter of the mean-field models of the glass transition in the presence of the randomly pinned spins/atoms at their dynamic transition lines [5–7]. As the fraction of the pinned spins, c , increases, the A_2 singular point $p_d(c)$, at which the order parameter discontinuously jumps, decreases (or $T_d(c)$ increases) and terminates at a larger but finite $c = c_s$, where the transition point becomes the higher order singular point of type A_3 , which is fully consistent with the dynamic transition lines predicted by the MCT for randomly pinned systems. The concept of the point-to-set correlation has been employed to evaluate the correlation length, ξ , around this transition line, $p_d(c)$ [27, 31]. Aside from the trivial factor of 2, the exponent, ν , of the correlation lengths, $\xi \sim \varepsilon^{-\nu}$, is consistent with that obtained from the inhomogeneous version of the MCT [9]. The exponent at the A_3 singularity point is $\nu_\perp = 1/3$ and $\nu_\parallel = 1/2$, depending on the directions to approach the critical point. This behavior is akin to that of the ferromagnetic transition in the Landau theory ($\nu = 1/2$ when the magnetic field $h = 0$ and $T \rightarrow T_c$ and $\nu = 1/3$ when $T = T_c$ and $h \rightarrow 0$ [9]). Note that the parallel direction is more special than the perpendicular direction, because the critical behavior along the former is observed only when $\varepsilon_\perp = 0$, whereas the later is generic in a sense that the critical behavior is observed even if $\varepsilon_\perp \neq 0$. We should emphasize, however, that the analysis of the order parameter as studied here can not establish the universality class of the transition. It is known that the transition of the mean-field random-field Ising model (RFIM) is also characterized by the same exponents as those of the ferromagnetic transition in the Landau theory but its universality class is distinct [32]. The analysis

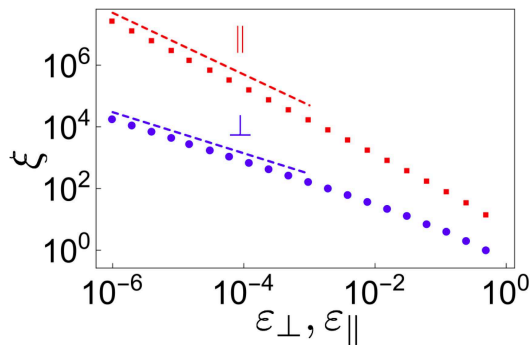


Fig. 7: ξ near the terminal critical point ($c_s = 1/5$, $p_s = 5/6$) approached from the parallel (\perp) and perpendicular (\parallel) directions. Numerical results are shown in filled symbols. Thin dashed lines represent $|\varepsilon_{\perp}|^{-2/3}$ and $|\varepsilon_{\parallel}|^{-1}$, respectively.

of the fluctuations such as the higher order moments is necessary to characterize the nature of the transition. Indeed, it has been demonstrated that from the dynamic scaling of the model discussed here at $c = 0$, the model belongs to the RFIM universality class [33]. Besides, the randomly pinned glass model is also argued to belong to the same universality class [8].

In this letter, we only analyzed the stationary properties of the randomly pinned FAM. It is known that the slow relaxation dynamics of the FAM on the Bethe lattice for $c = 0$ is qualitatively similar to that of the MCT for the bulk supercooled liquid or mean-field spin glass models [16]. On the other hand, the MCT and IMCT predict that the slow dynamics near the A_3 singularity point is distinct from that at the A_2 point, and the system exhibits anomalous dynamics such as the logarithmic relaxation and milder growth of the correlated fluctuations [3, 9]. Studies in this direction are left for future works.

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